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On the Computation of Eigenvalues Arising out of Perturbations of the Blasius Profile

G. WILKS AND J. S. BRAMLEY

Department of Mathematics, University of Strathclyde, Glasgow, Scotland

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An examination of the application of the Riccati transformation method to an odd order differential system over a semi-infinite interval is described. An inherent flexibility of the method is exploited to yield a formulation which proves to be efficient, accurate, and straightforward in application. The present limitations of the method for general use on this type of problem are outlined.

1. INTRODUCTION

The Riccati transformation method for the computation of eigenvalues of a system of linear ordinary differential equations was first introduced by Scott [1]. He considered a system of the form

$$\begin{aligned} d\mathbf{u}/dz &= A(z, \sigma)\,\mathbf{u} + B(z, \sigma)\,\mathbf{v}, \\ -(d\mathbf{v}/dz) &= C(z, \sigma)\,\mathbf{u} + D(z, \sigma)\,\mathbf{v}, \end{aligned} \tag{1}$$

subject to the linear separated boundary conditions

$$\mathbf{u}(0) = \mathbf{0}, \qquad \mathbf{u}(x) = \mathbf{0}, \qquad \text{or} \qquad \mathbf{v}(x) = \mathbf{0}. \tag{2}$$

Here $\mathbf{u}$ and $\mathbf{v}$ are n-vectors and $A(z, \sigma)$, $B(z, \sigma)$, $C(z, \sigma)$, and $D(z, \sigma)$ are $n \times n$ real matrices which depend on the independent variable z and some scalar parameter σ. Scott described the evaluation of eigenvalues via the related problem of calculating characteristic lengths of the system. A characteristic length of the system is defined to be a positive value $z = x$ for which a nontrivial solution exists when σ takes a prescribed value. For specified x, an eigenvalue is obtained by establishing in an interpolative manner, that σ for which x is a characteristic length. The method locates characteristic lengths by transforming (1) to an associated system of nonlinear Riccati differential equations which are integrated from $z = 0$ using known initial conditions on all elements. As the integration of this initial-value problem proceeds, characteristic lengths are located by applying a simple criterion to the elements of the Riccati system.

Sloan and Wilks [2] extended the above method to be used with general separated boundary conditions with n conditions at either end of a finite interval. It is natural

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to inquire into the possibility of extending the method to deal with problems, with $n > 1$, contravening these specifications. In particular one might consider the possible application to eigenvalue problems involving infinite or semi-infinite intervals or involving odd ordered systems. Scott [3] solved an eigenvalue problem on a semi-infinite interval for the scalar case $n = 1$ and Alspaugh [4] used the Riccati transformation to solve linear two-point boundary-value problems also on an infinite interval. Equation (1) and boundary conditions (2) form a basis for work in subsequent chapters of this paper and salient features from [1, 2, 3] are outlined in the next chapter.

By examining a problem already well documented in the literature of perturbation analysis in boundary layer theory, the work that follows illustrates the possibilities of extension of the method to such circumstances. The problem, that of computing the eigenvalues arising out of perturbations of the Blasius profile, involves, however, not only an odd order system over a semi-infinite interval but also incorporates difficulties encountered in isolating eigenvalues associated with exponential decay. Consequently a model equation exhibiting analogous behavior is first examined to test the efficacy of the method with respect to this additional difficulty. Subsequently various investigations with respect to the title problem are described which highlight the possibility of optimizing an inherent flexibility of the method. The final formulation confirms that the method may indeed be used successfully to obtain eigenvalues for this type of problems which so often occurs in boundary layer theory.

2. An Outline of the Method

The method described by Scott [1] for the problem defined by (1) and (2) is based on the introduction of an $n \times n$ matrix $R(z)$ by means of the transformation

$$\mathbf{u}(z) = R(z)\,\mathbf{v}(z). \tag{3}$$

It is readily shown that if R satisfies the matrix Riccati equation

$$R'(z) = B(z, \sigma) + A(z, \sigma)\,R(z) + R(z)\,D(z, \sigma) + R(z)\,C(z, \sigma)\,R(z) \tag{4}$$

and if $\mathbf{v}(z)$ is a solution vector then the associated solution vector $\mathbf{u}(z)$ is given by (3). Here the superposed dash denotes d/dz. The condition $\mathbf{u}(0) = \mathbf{0}$ permits the integration of (4) using the initial condition $R(0) = 0$. Characteristic lengths are values $z = x$ where $\mathbf{u}(x) = \mathbf{0}$, and this terminating condition is only satisfied when

$$\det R(x) = 0. \tag{5}$$

In the course of the integration there may be points at which $\det R(z)$ is singular and these are traversed by switching to $S(z) = R^{-1}(z)$ which satisfies

$$-S'(z) = C(z, \sigma) + S(z)\,A(z, \sigma) + D(z, \sigma)\,S(z) + S(z)\,B(z, \sigma)\,S(z). \tag{6}$$

Values of z at which $\det S(z) = 0$ are characteristic lengths for the problem with terminating condition $\mathbf{v}(x) = \mathbf{0}$.

Scott's work was extended by Sloan and Wilks [2] to deal with the general linear separated boundary conditions

$$\alpha_1\mathbf{u}(0) + \beta_1\mathbf{v}(0) = \mathbf{0}, \qquad \alpha_2\mathbf{u}(x) + \beta_2\mathbf{v}(x) = \mathbf{0}, \tag{7}$$

where the real matrices $[\alpha_1, \beta_1]$ and $[\alpha_2, \beta_2]$ have dimensions $n \times 2n$ and rank n. The vectors $\mathbf{u}$ and $\mathbf{v}$ may be chosen in such a way that $\alpha_2 = I$ and $\beta_2 = 0$, in which case the boundary conditions take the form

$$\alpha_1\mathbf{u}(0) + \beta_1\mathbf{v}(0) = \mathbf{0}, \qquad \mathbf{u}(x) = \mathbf{0}. \tag{8}$$

Sloan and Wilks tackled the problem (1) and (7) by suitably transforming the dependent variables. If

$$\begin{bmatrix}\mathbf{U}(z)\\ \mathbf{V}(z)\end{bmatrix} = \begin{bmatrix}\alpha_1 & \beta_1\\ \gamma_1 & \delta_1\end{bmatrix}\begin{bmatrix}\mathbf{u}(z)\\ \mathbf{v}(z)\end{bmatrix} = M\begin{bmatrix}\mathbf{u}(z)\\ \mathbf{v}(z)\end{bmatrix}, \tag{9}$$

with γ_1 and δ_1 chosen such that M is nonsingular, then the condition at $z = 0$ takes the form

$$\mathbf{U}(0) = \mathbf{0}. \tag{10}$$

The possibility of dealing with problems with separated boundary conditions and with $n = 1$ using a linear transformation of dependent variables is mentioned in the book by Scott [3]. Relating $\mathbf{U}$ and $\mathbf{V}$ by

$$\mathbf{U}(z) = E(z)\,\mathbf{V}(z) \tag{11}$$

we readily derive the Riccati equation

$$E'(z) = \mathscr{B}(z, \sigma) + \mathscr{A}(z, \sigma)\,E(z) + E(z)\,\mathscr{D}(z, \sigma) + E(z)\,\mathscr{C}(z, \sigma)\,E(z). \tag{12}$$

The condition $\mathbf{U}(0) = \mathbf{0}$ shows that $E(0) = \mathbf{0}$. Here $\mathscr{A}$, $\mathscr{B}$, $\mathscr{C}$, and $\mathscr{D}$ are coefficient matrices in the linear differential system when it is written in terms of $\mathbf{U}$ and $\mathbf{V}$.

If the original vectors $\mathbf{u}$ and $\mathbf{v}$ have been chosen such that the boundary conditions are of type (8), the condition at $z = x$ is then similar to that considered by Scott [1] and characteristic lengths may be located using criterion (5). In order to use this criterion a switch is made from the E-system to the R-system at some point $z = \bar{z} > 0$ using the transformation

$$R(\bar{z}) = [\alpha_1 - E(\bar{z})\gamma_1]^{-1}[E(z)\,\delta_1 - \beta_1]. \tag{13}$$

Note that at $z = \bar{z}$ the matrix $[\alpha_1 - E(\bar{z})\,\gamma_1]$ must be nonsingular. The method adopted in this paper for the location of characteristic lengths is that outlined above for a system (1) subject to boundary conditions (8).

The procedure for obtaining characteristic lengths is now

(i) initiate integration of the E-system of Eqs. (12) using

$$E(0) = 0;$$

(ii) at a suitable point away from singularities of $[\alpha_1 - E(\bar{z})\,\gamma_1]$ switch to the R-, S-scheme using (13);

(iii) proceed as before to locations at which

$$\det R(x) = 0.$$

One could, of course, perform the complete integration in terms of the E-system, traversing singularities in $\det E$ by means of a complementary system involving E^{-1}. In this case the terminating condition could be expressed in terms of the E-components. This procedure is illustrated in the accompanying paper by Sloan [5].

3. An Example

In the Introduction it was noted that the isolation of eigenvalues associated with exponential decay provides a particular difficulty in examinations of perturbations about the Blasius profile. Accordingly the method is first examined in a context exhibiting analogous behavior. The equation

$$y''(z) + zy'(z) + \sigma y(z) = 0 \tag{14}$$

subject to the boundary conditions

$$y(0) = 0; \qquad y(z) \to 0 \quad \text{as } z \to \infty \tag{15}$$

has a continuous spectrum of eigenvalues for $\sigma > 0$. Solutions associated with this spectrum however exhibit algebraic decay at infinity. If only those solutions exhibiting exponential decay are acceptable then the discrete set of eigenvalues $\sigma = 2, 4, 6, \ldots, 2n, \ldots$ is appropriate.

It is interesting to speculate as to what one might anticipate of the application of the method over a semi-infinite interval. In an example with a finite interval, $z = 0$ to $z = x$, the typical pattern of events would, as for any initial-value method, essentially be an assembly of information generating curves $\sigma = \sigma_i(x)$ as in Fig. 1.

Eigenvalues associated with a particular value $x = L$ may then readily be extracted by interpolation. What one might expect therefore of curves $\sigma_i(x)$ in a semi-infinite context is a behavior asymptotic to the desired eigenvalues as $x \to \infty$. An investigation of this conjecture is readily performed by solving Eq. (14) subject to

$$y(0) = 0; \qquad y(x) = 0. \tag{16}$$

If Eq. (14) is represented as the pair of first-order equations

$$\begin{aligned} y_1'(z) &= y(z), \\ y_2'(z) &= -zy_2(z) - \sigma y_1(z), \end{aligned} \tag{17}$$

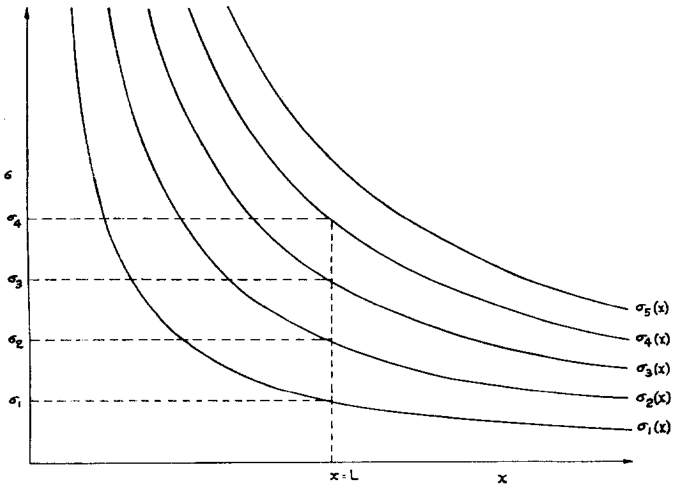

FIGURE 1

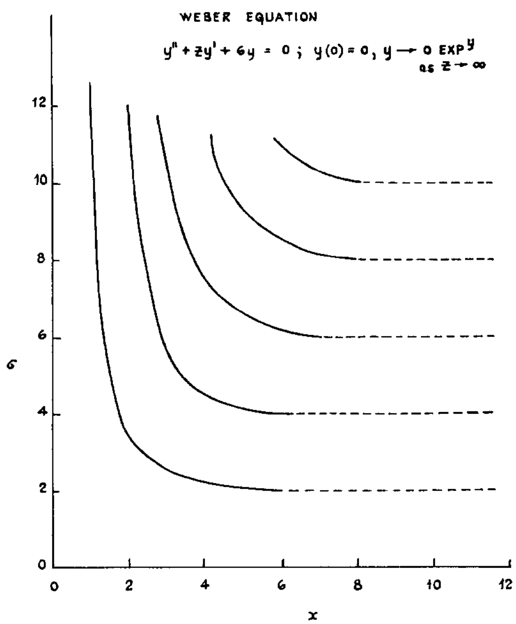

FIG. 2. Exponential decay eigenvalues.

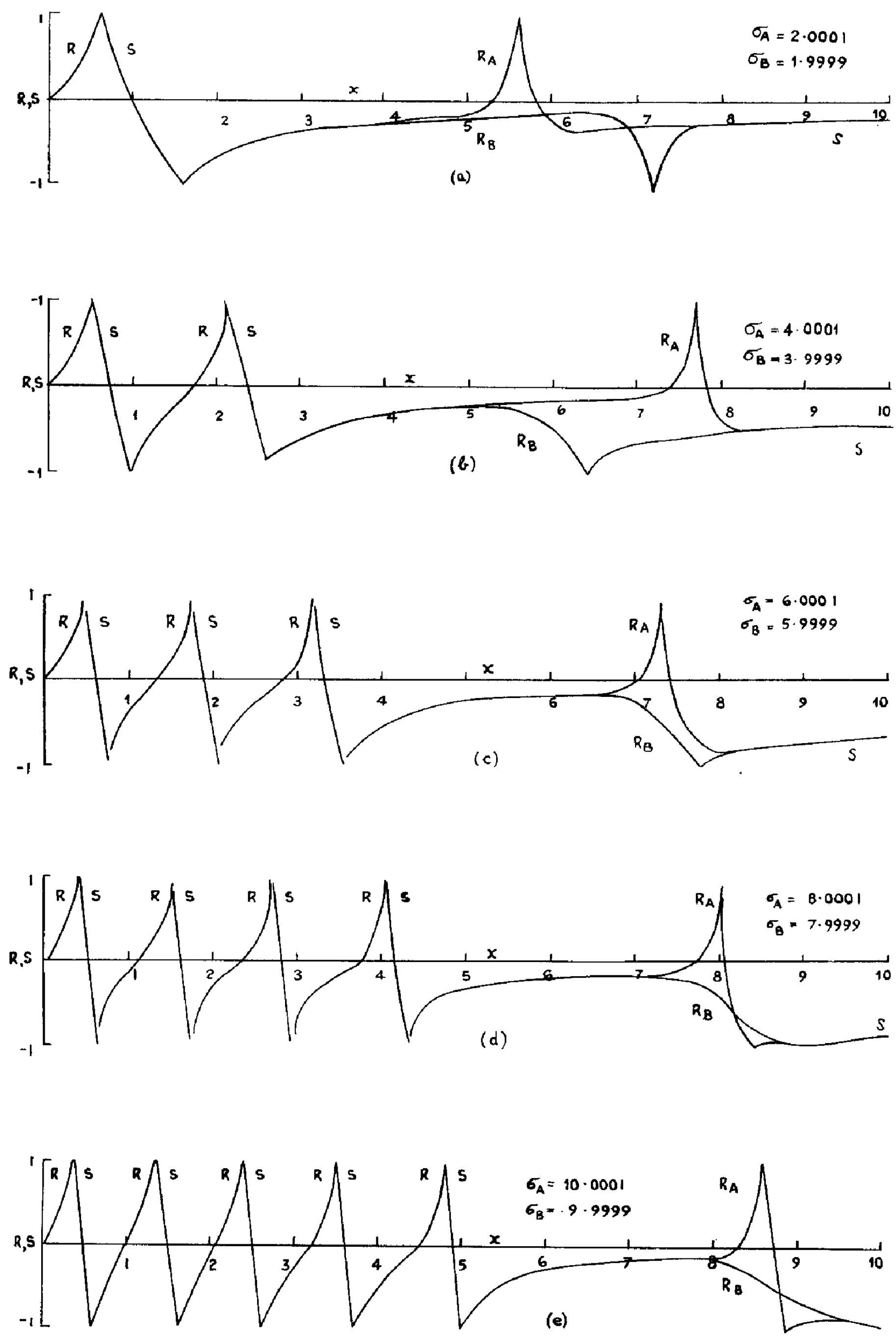

FIG. 3. R, S schemes exhibiting $\sigma = 2, 4, 6, 8, 10$ as demarcation values for zeros of R.

then in the notation of the method outlined in Section 2 we have

$$y_1(z) = u(z); \qquad y_2(z) = v(z) \tag{18}$$

and

$$\begin{aligned} (du/dz)(z) &= 0 \cdot u(z) + 1 \cdot v(z), \\ -(dv/dz)(z) &= \sigma \cdot u(z) + z \cdot v(z), \end{aligned} \tag{19}$$

so that

$$A(z, \sigma) = 0, \qquad B(z, \sigma) = 1, \qquad C(z, \sigma) = \sigma, \qquad D(z, \sigma) = z.$$

The associated Riccati equation and its complement are thus

$$R'(z) = 1 + zR(z) + \sigma R(z)^2, \tag{20}$$

$$S'(z) = -\sigma - zS(z) - S(z)^2. \tag{21}$$

Integration of (20) is initiated using $R(0) = 0$ and proceeds, switching to (21) when necessary, establishing characteristic lengths x where

$$R(x) = 0.$$

The simplicity of formulation and implementation is self-evident. The results of integrations for various σ using a standard fourth-order Runge–Kutta procedure with step length 0.05 and tolerance 1×10^{-6} are presented in Fig. 2. Accuracy to four decimal places, at least, was obtained for eigenvalues 2, 4, 6, 8, and 10 with no signs of instability or degeneracy for higher values of σ. In Fig. 3 curves of $R(z)$ and $S(z)$ for two values of σ displaced by 1×10^{-4} about the discrete set 2, 4, 6, 8, 10 demonstrate the implications of the search for locations $R(x) = 0$ for which nontrivial solutions exist.

The success of the method in isolating the discrete set of eigenvalues associated with exponential decay is thus exemplified. It is interesting to note the similarity between this procedure and that suggested by Stewartson [6] for obtaining unique solutions of the Falkner–Skan equations for $\beta < 0$. Solving over a finite interval and letting that interval tend to infinity is very much in the spirit of that author's remarks on selecting the acceptable solutions out of a semi-infinite family of solutions satisfying the boundary conditions.

4. The Problem

The mathematical problem is to find eigenvalues σ_n which give rise to nontrivial solutions of the homogeneous ordinary differential equation

$$y'''(z) + f_0(z)\, y''(z) + \sigma f_0'(z)\, y'(z) + (1 - \sigma)\, f_0''(z)\, y(z) = 0 \tag{22}$$

subject to the homogeneous boundary conditions

$$y(0) = y'(0) = 0; \qquad y'(z) \to 0 \text{ exponentially as } z \to \infty. \tag{23}$$

Dashes denote derivatives with respect to the independent variable z and $f_0(z)$ is the Blasius solution for the uniform flow of a viscous, incompressible flow past a semi-infinite plate defined by

$$f_0'''(z) + f_0(z) f_0''(z) = 0 \tag{24}$$

with

$$\begin{gathered} f_0(0) = f_0'(0) = 0; \qquad f_0'(\infty) = 1, \\ (f_0''(0) = 0.469600). \end{gathered} \tag{25}$$

The problem is of interest in two contexts. The first is based on a demonstration by Libby and Fox [7] that eigenfunctions of (22) form a complete orthogonal set with respect to functions having exponential decay at infinity. In this context the orthogonality property is exploited and an approximation technique, involving series expansions in terms of the complete set of eigenfunctions, is developed to deal with a variety of flow situations including heat and mass transfer [7, 8]. Although the first few eigenvalues were estimated in [7] the authors acknowledged that their method for obtaining these degraded in accuracy as successively higher eigenvalues are sought. Libby [9] returned to the problem with a view to obtaining an extended set of eigenvalues and eigenfunctions having noted that in application the approximation technique involved slowly convergent series and the consequent need for further terms. An improved method of computing the required eigenvalues was presented using backward integration based on asymptotic analysis. These extended results are taken to be the most accurate available at the present time. The agreement between these estimates and those forecast by Brown [10] after improving an asymptotic expansion for higher eigenvalues presented by Stewartson [11] supports this belief.

The second context is associated with higher order boundary layer theory. In coordinate asymptotic expansions about the Blasius solution the locations of the first few eigenvalues play an important part in appreciating the correct form of such an asymptotic expansion.

5. Preliminary Investigations

Having demonstrated that the method may be expected to deal with the semi-infinite interval of the problem and successfully isolate exponential decay eigenvalues it remains to accommodate the odd order of the problem within the even order formulation of the method. Since f_0' is a complementary solution, but not an eigenfunction, of Eq. (22) this equation may be reduced to a second-order differential equation. This reduced equation, however, is not amenable to Riccati formulation and accordingly as a first step we examine the even order equation obtained from a single differentiation of Eq. (22) namely,

$$y^{\text{iv}}(z) + f_0(z)\, y'''(z) + f_0'(z)(1+\sigma)\, y''(z) + f_0''(z)\, y'(z) + (1-\sigma) f_0'''(z)\, y(z) = 0. \quad (26)$$

The implementation of the method then requires the specification of a fourth boundary condition at a finite location $z = x$ which will, in the limit $x \to \infty$, yield appropriate behavior at infinity and also provide solutions pertinent to the original third-order system. The choice of this fourth boundary condition proves to be particularly significant in the course of extracting characteristic lengths from the associated systems.

I. *Approximate Boundary Conditions*

The Blasius solution of Eq. (24) subject to (25) has asymptotic behavior as $z \to \infty$ given by

$$f_0(z) \simeq z - c_1, \qquad f_0'(z) \simeq 1, \qquad f_0''(z) \simeq c_2 \exp[-(z - c_1)^2/2] \quad (27)$$

where

$$c_1 = 1.21676; \qquad c_2 = 0.33054.$$

For z sufficiently large so that

$$(z - c_1)^{-1} \exp[-(z - c_1)^2/2] \ll 3/c_2, \quad (28)$$

Eq. (22) takes the asymptotic form

$$y_\infty'''(z) + (z - c_1)\, y_\infty''(z) + \sigma y_\infty'(z) \simeq 0 \quad (29)$$

and

$$\begin{aligned} y_\infty'(z) \simeq{} & a_1(z - c_1)^{-(1-\sigma)} \exp[-(z - c_1)^2/2] + a_2(z - c_1)^{-\sigma} \\ & + a_3 \exp[-(z - c_1)^2/2] \end{aligned} \quad (30)$$

where a_1, a_2, a_3 are arbitrary constants. The isolation of solutions exhibiting exponential decay is achieved if

$$0 \simeq y_\infty''(z) + (z - c_1)\, y_\infty'(z)[1 + (1-\sigma)(z - c_1)^{-2}] \simeq a_2 \quad (31)$$

is satisfied. In view of these features for large z two approximate fourth boundary conditions were examined

$$\left.\begin{aligned} &\text{(a)} & y''(x) &= 0, \\ &\text{(b)} & y'''(x) + (x - c_1)\, y''(x) + \sigma y'(x) &= 0. \end{aligned}\right\} \quad (32)$$

With Eq. (28) represented as the set of four first-order differential equations

$$
\begin{aligned}
y_1'(z) &= y_2(z), \\
y_2'(z) &= y_3(z), \\
y_3'(z) &= y_4(z), \\
y_4'(z) &= -f_0(z)\, y_4(z) - f_0'(z) \cdot (1 + \sigma)\, y_3(z) - f_0''(z)\, y_2(z) - (1 - \sigma) \cdot f_0'''(z)\, y_1(z)
\end{aligned} \tag{33}
$$

an E-system appropriate to the initiation procedure of [2] and suitable for either condition (a) or (b) is obtained from a formulation with

$$\mathbf{U} = \begin{pmatrix} U_1 \\ U_2 \end{pmatrix} = \begin{pmatrix} y_1(z) \\ y_2(z) \end{pmatrix}; \qquad \mathbf{V} = \begin{pmatrix} V_1 \\ V_2 \end{pmatrix} = \begin{pmatrix} y_3(z) \\ y_4(z) \end{pmatrix}$$

as

$$
\begin{aligned}
E_1' &= E_3 + E_2 f_0' \cdot (1 + \sigma) + E_1 E_2 f_0''' \cdot (1 - \sigma) + E_2 E_3 f_0'', \\
E_2' &= E_4 - E_1 + E_2 f_0 + E_2{}^2 \cdot f_0''' \cdot (1 - \sigma) + E_2 E_4 f_0'', \\
E_3' &= 1 + E_4 \cdot f_0' \cdot (1 + \sigma) + E_1 E_4 \cdot f_0''' \cdot (1 - \sigma) + E_3 E_4 f_0'', \\
E_4' &= -E_3 + E_4 f_0 + E_2 E_4 \cdot f_0'''(1 - \sigma) + E_4^2 f_0''.
\end{aligned} \tag{34}
$$

An integration of this system commencing from

$$E_i(0) = 0 \qquad (i = 1, 2, 3, 4)$$

exhibits no singularities over the range of σ examined. This is particularly convenient in view of the correlation between E and R given by (13), which can be inverted to read

$$E(z) = [\alpha_1 R + \beta_1][\gamma_1 R + \delta_1]^{-1} \tag{35}$$

in the notation of Section 2.

Since the R, S formulation would require

$$\mathbf{u} = \begin{pmatrix} & y_2 \\ \text{(a)} & y_3 \\ \text{(b)} & y_4 + (z - c_1)\, y_3 + k y_2 \end{pmatrix}; \qquad \mathbf{v} = \begin{pmatrix} y_1 \\ y_4 \end{pmatrix}$$

the correlations between $(\mathbf{u}, \mathbf{v})$ and $(\mathbf{U}, \mathbf{V})$ are given by

$$
\begin{aligned}
&\text{(a)} \quad \alpha_1 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}; \quad \beta_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}; \quad \gamma_1 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}; \quad \delta_1 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \\
&\text{(b)} \quad \alpha_1 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}; \quad \beta_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}; \quad \gamma_1 = \begin{pmatrix} 0 & 0 \\ -\sigma & 1 \end{pmatrix}; \quad \delta_1 = \begin{pmatrix} 0 & 1 \\ 0 & -(z - c_1) \end{pmatrix}
\end{aligned} \tag{36}
$$

which in (35) yield

$$\text{(a)} \qquad E(z) = \frac{1}{R_3}\begin{pmatrix} 1 & -R_4 \\ R_1 & -\det R \end{pmatrix},$$
$$\text{(b)} \qquad E(z) = \frac{1}{R_3 - \sigma R_1}\begin{pmatrix} \sigma R_2 - R_4 + (z - c_1) & 1 \\ R_1(z - c_1) - \det R & R_1 \end{pmatrix}. \tag{37}$$

The implications are clear. Since the E-system (34) displays no singularities, monitoring the locations at which

$$\text{(a)} \qquad E_4(z) = 0 \qquad (R_3 \neq 0),$$
$$\text{(b)} \qquad E_3(z) - (z - c_1)\, E_4 = 0 \qquad (R_3 - \sigma R_1 \neq 0), \tag{38}$$

is precisely equivalent to monitoring the respective criterion for characteristic lengths, $\det R(x) = 0$. No recourse to the R-, S-system is required. On examining this feature results were obtained entirely in agreement with the conjecture of Section 3. However, close inspection of the asymptotes evolving displayed a degradation of accuracy in estimates of successively higher eigenvalues. See Table I.

TABLE I

	σ_n						
				Wilks and Bramley			
n	Libby and Fox	Libby	Brown	I(a)	I(b)	II Exact	General exact
1	2.000	2.000		2.000	2.000	2.000	2.0000
2	3.774	3.774		3.775	3.774	3.774	3.7736
3	5.635	5.629		5.632	5.630	5.629	5.6287
4	7.600	7.513		7.530	7.520	7.515	7.5132
5	9.480	9.414					9.4144
6	11.3	11.327					11.3265
7	13.2	13.247					13.2467
8	15.1	15.173					15.1731
9	16.9	17.104					17.1044
10	18.7	19.040					19.0397
11		20.979					20.9784
12		22.920					22.9201
13		24.865					24.8642
14		26.811					26.8107
15		28.760					28.7591
16		30.710					30.7093
17		32.662	32.660				32.6611
18		34.615	34.613				34.6143
19		36.570	36.567				36.5689
20		38.526	38.524				38.5247

II. *An Exact Boundary Condition*

If the general solution of Eq. (22) is denoted by $Y^*(z)$ then the fact that Eq. (26) is the first derivative of Eq. (22) requires that the general solution $Y(z)$ of Eq. (26) must be

$$Y(z) = Y^*(z) + cg(z) \tag{39}$$

where c is an arbitrary constant and $g(z)$ is any solution of the inhomogeneous form of Eq. (22)

$$y'''(z) + f_0(z)\,y''(z) + \sigma f_0'(z)\,y'(z) + (1-\sigma)\,f_0''(z)\,y(z) = 1 \tag{40}$$

or

$$L(y(z)) = 1, \tag{41}$$

say. Then at any point x

$$L[Y(x)] = L[Y^*(x)] + cL[g(x)] = c. \tag{42}$$

The implication must be that equivalence of the even order formulation with the original problem is achieved by incorporating as additional boundary condition

$$L[Y(x)] = 0$$

that is

$$y'''(x) + f_0(x)\,y''(x) + \sigma f_0'(x)\,y'(x) + (1-\sigma)\,f_0''(x)\,y(x) = 0. \tag{43}$$

Equation (43) is thus deemed to be an exact supplementary boundary condition.

With

$$\mathbf{U} = \begin{pmatrix} y_1(z) \\ y_2(z) \end{pmatrix}; \quad \mathbf{V} = \begin{pmatrix} y_3(z) \\ y_4(z) + f_0(z)\,y_3(z) + \sigma f_0'(z)\,y_2(z) + (1-\sigma)\,f_0''(z)\,y_1(z) \end{pmatrix}, \tag{44}$$

$$\mathscr{A} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}; \quad \mathscr{B} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}; \quad \mathscr{C} = \begin{pmatrix} f_0'' & (1-\sigma) & \sigma f_0' \\ & 0 & 0 \end{pmatrix}; \quad \mathscr{D} = \begin{pmatrix} f_0 & -1 \\ 0 & 0 \end{pmatrix}, \tag{45}$$

and the appropriate Riccati initiation E-system is

$$\begin{aligned}
E_1' &= E_3 + E_1 \cdot f_0 + E_1^2 \cdot f_0'' \cdot (1-\sigma) + E_1 E_3 \cdot \sigma \cdot f_0', \\
E_2' &= E_4 - E_1 + E_1 E_2 \cdot f_0'' \cdot (1-\sigma) + E_1 E_4 \cdot \sigma \cdot f_0', \\
E_3' &= 1 + E_3 \cdot f_0 + E_1 E_3 \cdot f_0'' \cdot (1-\sigma) + E_3^2 \cdot \sigma \cdot f_0', \\
E_4' &= -E_3 + E_2 \cdot E_3 \cdot f_0'' \cdot (1-\sigma) + E_3 E_4 \cdot \sigma \cdot f_0',
\end{aligned} \tag{46}$$

with, again

$$E_i(0) = 0 \qquad (i = 1, 2, 3, 4). \tag{47}$$

Outward integration of this system *is limited by a singularity* and recourse to the R-, S-system is required. After setting

$$\mathbf{u} = \begin{pmatrix} (y_2 z) \\ y_4(z) + f_0(z)\, y_3(z) + \sigma f_0'(z) \cdot y_2(z) + (1-\sigma) f_0''(z) \cdot y_1(z) \end{pmatrix}; \quad \mathbf{v} = \begin{pmatrix} y_1(z) \\ y_3(z) \end{pmatrix}, \tag{48}$$

$$A = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}; \quad B = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}; \quad C = \begin{pmatrix} -1 & 0 \\ \sigma f_0' & -1 \end{pmatrix}; \quad D = \begin{pmatrix} 0 & 0 \\ (1-\sigma) \cdot f_0'' & f_0 \end{pmatrix}, \tag{49}$$

the R-, S-systems are

$$\begin{aligned} R_1' &= (1-\sigma) \cdot f_0'' R_2 - R_1^2 + \sigma f_0' \cdot R_1 R_2 - R_2 R_3 , \\ R_2' &= 1 + f_0 R_2 - R_1 R_2 + \sigma f_0' \cdot R_2^2 - R_2 R_4 , \\ R_3' &= f_0'' \cdot (1-\sigma) \cdot R_4 - R_1 R_3 + \sigma f_0' R_1 R_4 - R_3 R_4 , \\ R_4' &= f_0 \cdot R_4 - R_2 R_3 + \sigma f_0' \cdot R_2 R_4 - R_4^2, \end{aligned} \tag{50}$$

and

$$\begin{aligned} S_1' &= 1 - S_1 S_3 , \\ S_2' &= -S_1 S_4 , \\ S_3' &= -\sigma \cdot f_0' - S_1 \cdot f_0'' \cdot (1-\sigma) - f_0 \cdot S_3 - S_3^2, \\ S_4' &= 1 - S_2 \cdot f_0'' \cdot (1-\sigma) - f_0 S_4 - S_3 S_4 . \end{aligned} \tag{51}$$

When the procedure outlined in Section 2 is implemented in the search for characteristic lengths, x, given by

$$\det R(x) = 0 \tag{52}$$

results entirely in keeping with the conjecture of Section 3 are again obtained. Moreover, the degeneracy in accuracy is no longer in evidence and precise estimates of the first four eigenvalues were obtained as in Table I. However this first attempt at an exact solution is hindered by another phenomenon which occurs as successively higher eigenvalues are sought. Although it is accepted that singularities in the E-, R-, S-systems are likely to occur, which the switching procedure normally accommodates, if a zero of $\det R(x) = 0$ approaches coincidence with such a singularity then the isolation of x becomes difficult. This tendency to coincidence of a zero and a singularity is apparent in the search for the fourth eigenvalue and immediately after the location of the fourth zero the equations become unintegrable.

These preliminary investigations are instructive. The experience with the approximate boundary conditions highlights the fact that although the machinery exists within the method to accommodate singularities in the associated systems via switching, this machinery need not necessarily be invoked. The implication from the exact boundary condition examination is that the most obvious exact boundary condition need not necessarily be the most efficient. There is advantage to be gained in exploiting the inherent flexibility of the method, either in the selection of vectors of dependent variables or equivalently in the method of rendering even the original odd order system. Accordingly the following section outlines a generalized exact formulation of the problem out of which an optimal procedure arises incorporating a nonsingular R-system.

6. The Generalized Exact Formulation

In Section 5 investigations were based on the fourth-order system generated by differentiating the original equations. This not the only means of evening up the system. In this section we illustrate a manipulation of the inherent flexibility within the method of the choice of vectors of dependent variables. An optimum choice in terms of computational advantage becomes apparent.

Consider the original Eq. (22) as the system of three first-order differential equations

$$\begin{aligned} y_1' &= y_2 , \\ y_2' &= y_3 , \\ y_3' &= -f_0 \cdot y_3 - \sigma f_0' \cdot y_2 - (1-\sigma) f_0'' \cdot y_1 , \end{aligned} \tag{53}$$

and augment the system with the dummy equation for a function $p(z)$ such that

$$p'(z) = y_4'(z) = 0; \qquad p(x) = 0. \tag{54}$$

An E-system preserving initiation $E_i(0) = 0$ is obtained after setting

$$\mathbf{U} = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}; \qquad \mathbf{V} = \begin{pmatrix} y_3 \\ y_4 \end{pmatrix}.$$

However it may be shown that the solution of this system with $E(0) = 0$ has the elements E_2 and E_4 identically zero. Since the matrix $[\alpha_1 - E(z)\gamma_1]$ is therefore singular it proves impossible to proceed to the R-, S-scheme of equations. Indeed a closer inspection of this simple augmentation reveals that it is impossible to generate two linearly independent solutions satisfying $\mathbf{U}(0) = 0$ and that any eigenfunction $\mathbf{U}$, of the system, under this initial condition can only be a multiple of a single base vector $\mathbf{U}_1(z)$, say. This is no longer a restriction if a constant multiple of $p = y_4$ is added to any or all of the dependent variables y_1, y_2, y_3. In practice this is equivalent to an examination of the system

$$\begin{aligned} y_1' &= y_2 + k_1 y_4 , \\ y_2' &= y_3 + k_2 y_4 , \\ y_3' &= -f_0 y_3 - \sigma f_0' y_2 - (1-\sigma) f_0'' y_1 + k_3 y_4 , \\ y_4' &= 0. \end{aligned} \tag{55}$$

Provided that at least one member of the triad (k_1, k_2, k_3) is nonzero the associated E-system will yield $[\alpha_1 - E(z)\gamma_1]$ nonsingular.

Maintaining full generality the E-system is

$$\begin{aligned} E_1' &= E_3 + E_1 f_0 + E_1^2 \cdot (1-\sigma) f_0'' + E_1 E_3 \sigma f_0', \\ E_2' &= k_1 + E_4 + E_1 k_3 + E_1 E_2 \cdot (1-\sigma) f_0'' + E_1 E_4 \sigma f_0', \\ E_3' &= 1 + E_3 f_0 + E_1 E_3 \cdot (1-\sigma) f_0'' + E_3^2 \sigma f_0', \\ E_4' &= k_2 + E_3 k_3 + E_2 E_3 \cdot (1-\sigma) f_0'' + E_3 E_4 \sigma f_0', \end{aligned} \tag{56}$$

and

$$E_i(0) = 0; \qquad i = 1, 2, 3, 4.$$

The flexibility of the system and the implications with respect to the E, R, S scheme have been investigated by allocating the value unity to each k_i in turn, the net effect being independent of magnitude or sign of a nonzero k_i. With the triads (1, 0, 0) and (0, 0, 1) exactly the same difficulty arises as was outlined for the exact boundary condition in Section 5. The R-system becomes unintegrable after a zero in det R. Indeed the triad (0, 0, 1) coincides with the exact boundary condition formulation of Section 5. However, on employing the triad (0, 1, 0) the related R-system displays no singularities and the monitoring of zeros of det R proves to be particularly straightforward.

7. Results and Discussion

Once the particular advantages of the triad (0, 1, 0) becomes apparent, attention is naturally focused on the use of the appropriate system out of Eq. (56) and the associated R-system

$$\begin{aligned} R_1' &= R_3 + R_2 \cdot (1-\sigma) f_0'' + \sigma f_0' R_1 R_2 - R_1^2, \\ R_2' &= 1 + R_4 + f_0 R_2 + \sigma f_0' R_2^2 - R_1 R_2, \\ R_3' &= R_4 \cdot (1-\sigma) f_0'' + \sigma f_0' R_1 R_4 - R_1 R_3, \\ R_4' &= f_0 R_4 + \sigma f_0' R_2 R_4 - R_2 R_3. \end{aligned} \tag{57}$$

In particular it is to be established whether or not calculation of higher eigenvalues will lead to any degeneracy of accuracy or further integration problems. In an investigation of these questions no complications in fact arise. The only noteworthy feature occurring as estimates for higher eigenvalues are sought concerns the behavior of det R which turns out to oscillate with decreasing amplitude as z increases. Consequently as higher eigenvalues are sought the number of oscillations of det R increases and amplitudes involved become extremely small, e.g., $O(10^{-20})$. However an examination of the elements of R reveals that R_1 and R_3 are of comparable order of magnitude, as also are R_1 and R_4. The use of specified relative error limits during integration of the R-system thus ensures no loss of accuracy in characteristic length location.

Accordingly we present in Table I what we believe to be accurate evaluations of the first twenty eigenvalues. The results of earlier workers are included for comparison. Of particular note is the favorable comparison of these numerical evaluations with asymptotic estimates of Brown [10].

The associated curves $\sigma_i(z)$ revealing the confirmation of the asymptotic conjecture are illustrated in Fig. 4. An interesting feature is that σ_1 and σ_2 originate from the same point, as do σ_3 and σ_4. This has implications in terms of the total zeros to be monitored for higher eigenvalue evaluations.

The investigations described in this paper have been of an exploratory nature and it would certainty be premature to conclude other than that the method has been made to work for this particular problem. Comparisons with other well established methods would be inappropriate at this stage. The final formulation used to obtain results is however very straightforward to implement, efficient in its use of computer time and storage and yields accurate results, and is accordingly attractive once arrived at. The

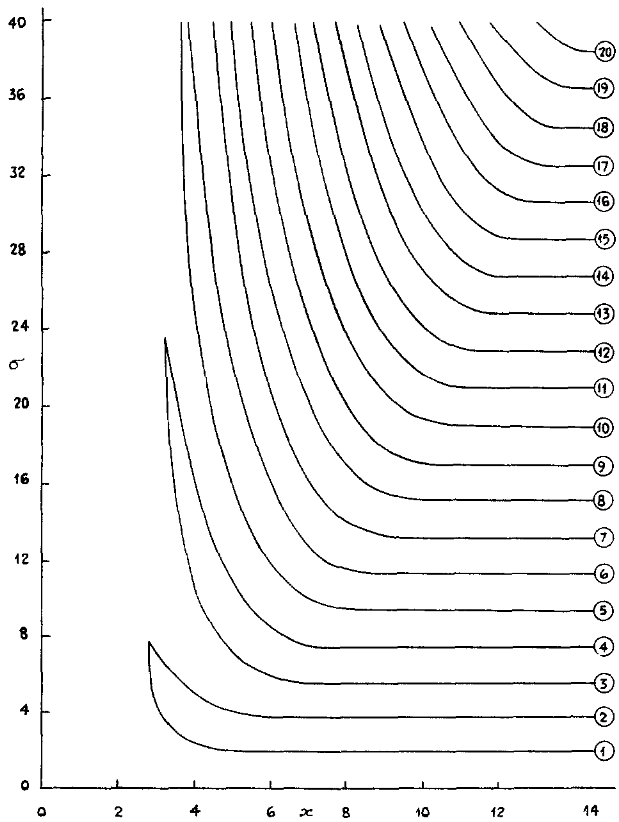

FIG. 4. Curves $\sigma_i(x)$ for Blasius profile.

problem outstanding is that of optimization for computational purposes of the choice of arrays of dependent variables. Should it be possible to select immediately an appropriate optimal choice for this type of problem then the method may indeed prove competitive with those currently in the literature.

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